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On Efficient Capital Accumulation in a Multi-Sector Neoclassical Model

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1. INTRODUCTION

The problems of characterizing efficient infinite programmes, and in particular efficient competitive programmes, have been much discussed in the literature (for a complete list of references, the interested reader should consult Cass [1], [2] and Majumdar [5]). For the standard neoclassical model with one producible good and a more general model with many capital goods and *one* consumption good, the fundamental result of Cass provides us with an easily applicable criterion for testing the efficiency of competitive programmes, viz. a competitive programme is efficient iff the terms of trade of the consumption good do not deteriorate too fast.

The present exercise can be thought of as a continuation of his work, and aims at deriving a complete characterization of efficient programmes in the neoclassical model of Dorfman-Samuelson-Solow [3]. The appropriate differentiability and curvature assumptions are imposed directly on the technology, and no restriction is imposed on the nature or number of consumption goods.

A difficulty that one faces, initially, in the many consumption good case is the following: " the terms of trade of which good should we use in formulating our criterion? " The answer is that we could use any good, by concentrating on competitive programmes. For the rates of transformation of consumption on all " processes " are equal along a competitive programme, and this would mean (for " interior " programmes) that the terms of trade of one good deteriorate too fast along a competitive programme iff the terms of trade of every good deteriorate too fast. Now, with appropriate adaptation of the arguments in Cass, a criterion analogous to his result can be derived—namely, an interior programme is efficient if and only if it is competitive and the terms of trade of none of the goods deteriorate too fast.

It should be noted that the result obtained here is qualitatively different from the transversality condition criterion obtained by Majumdar [5], for the simple linear model, with many consumption goods. The transversality condition criterion does not apply to our framework, just as our condition does not apply to the linear framework.

2. THE MODEL

Consider an economy in which there are m producible goods. The technology does not change over time and is described by a set $\mathscr{T}$ in the non-negative orthant of R^{2m}—a pair (x, y) is in $\mathscr{T}$ iff it is possible to get the output vector y in period $(t+1)$ from an input vector x in period t.

We denote the $(m-1)$ vectors $(y^1, \ldots, y^{m-1})$ and $(x^1, \ldots, x^{m-1})$ by $^*y$ and $^*x$ respectively. For any two n-vectors, u and v, $u \geqq v$ means $u^i \geqq v^i$ for all $i = 1, \ldots, n$; $u > v$ means $u \geqq v$ and $u \neq v$; $u \gg v$ means $u^i > v^i$ for all $i = 1, \ldots, n$.

The following assumptions are made on $\mathscr{T}$:

A.1. *$\mathscr{T} = \{(x, y)\colon 0 \leqq y^m \leqq f(^*y, x)\}$, where $f(\cdot)$ is a non-negative real-valued function on the non-negative orthant of R^{2m-1}, such that it is twice continuously differentiable for $(^*y, x) \gg 0$.*

A.2. *$\partial f/\partial y^i < 0$ for $i = 1, 2, ..., m-1$; $\partial f/\partial x^i > 0$ for $i = 1, 2, ..., m$ when the derivatives are evaluated at any $(^*y, x) \gg 0$.*

A.3. *$f(\cdot)$ is strictly concave in its $(2m-1)$ arguments and the matrix of second-order partial derivatives of $f(\cdot)$ is negative definite when evaluated at $(^*y, x) \gg 0$.*

It should be understood that $f(\cdot)$ gives the maximum value of the output of the mth good, given the values of its arguments.

We define a *feasible programme* from $\mathbf{x}$ as a sequence $(x, y, c) = (x_t, y_{t+1}, c_{t+1})$ such that $x_0 = \mathbf{x}$, $y^m_{t+1} = f(^*y_{t+1}, x_t)$ for all $t \geqq 1$; $c_{t+1} = y_{t+1} - x_{t+1}$, and $x_t, y_{t+1}, c_{t+1} \geqq 0$ for all $t \geqq 0$.

We next assume

A.4. *There exists a scalar $K > 0$, such that for any feasible programme from $\mathbf{x}$, $x^i_t, y^i_{t+1} \leqq K$, for all $t \geqq 0$ and $i = 1, ..., m$.* (K depends only on $\mathbf{x}$.)

This is traditionally either assumed in the literature (cf. McKenzie [4]), or proved by starting from a technology set, including labour as an essential primary factor for production (growing exponentially), and establishing the boundedness property on a normalized technology set (cf. Peleg and Ryder [6]).

A feasible programme $(\bar{x}, \bar{y}, \bar{c})$ from $\mathbf{x}$ is *short-run efficient* if there is no other feasible programme (x, y, c) from $\mathbf{x}$, such that $(c_1, ..., c_T, x_T) > (\bar{c}_1, ..., \bar{c}_T, \bar{x}_T)$ for some $T \geqq 1$. It is *long-run efficient* if there is no other feasible programme (x, y, c) from $\mathbf{x}$ with $c_t \geqq \bar{c}_t$ for all $t \geqq 1$, $c_t > \bar{c}_t$ for some t.

It is immediate, in our framework that a long-run efficient programme is necessarily short-run efficient. However, the converse is not true, i.e. a feasible programme such that every finite segment is efficient need not be long-run efficient. For a detailed discussion of these two concepts, see Cass [2].

We shall be restricting our characterization (in the next section) to programmes which are " interior ", i.e. which are bounded away from zero in all input levels. Formally, a feasible programme (x, y, c) from $\mathbf{x}$ is *interior* if there exists a scalar $k > 0$ such that $x^i_t \geqq k$, for all $t \geqq 0$, and $i = 1, 2, ..., m$.

Associated with an interior programme $(\bar{x}, \bar{y}, \bar{c})$ is a uniquely defined current and discounted *price system*. Denote

$$\bar{f}^{t+1}_{x^i} = \partial f(^*\bar{y}_{t+1}, \bar{x}_t)/\partial x^i, \quad i = 1, ..., m; \ \bar{f}^{t+1}_{y^i} = \partial f(^*\bar{y}_{t+1}, \bar{x}_t)/\partial y^i, \quad i = 1, ..., m-1; \ t \geqq 0$$

$$\bar{\pi}^m_t = \Pi_s \bar{f}^{s+1}_{x^m}, \ \bar{\pi}^i_t = \Pi_s \bar{f}^{s+1}_{x^i}/(-\bar{f}^s_{y^i}) \quad \text{for } i = 1, ..., m-1, \ t \geqq 2, \text{ and } s = 1, ..., t-1,$$

$$\bar{\pi}^i_1 = 1, \ \bar{\pi}^i_0 = 1, \ i = 1, ..., m.$$

The *current price sequence* $(\bar{q}) = (\bar{q}_t)$ is defined by:

$$\bar{q}^m_t = 1, \text{ and } \bar{q}^i_t = -\bar{f}^t_{y^i} \text{ for } i = 1, ..., m-1 (t \geqq 1) \bar{q}^i_0 = \bar{f}^1_{x^i} \quad \text{for } i = 1, ..., m.$$

The *discounted price sequence* $(\bar{p}) = (\bar{p}_t)$ is defined by:

$$\bar{p}^i_t = \bar{q}^i_t/\bar{\pi}^i_t \quad \text{for } i = 1. ..., m, \ t \geqq 0.$$

An interior programme (x, y, c) is *competitive* if it satisfies for $t \geqq 1$

$$-\bar{f}^t_{y^i} = \bar{f}^{t+1}_{x^i}/\bar{f}^{t+1}_{x^m} \quad i = 1, ..., m. \qquad ...(2.1)$$

It is easy to see that for an interior programme $(\bar{x}, \bar{y}, \bar{c})$ which is competitive,

$$\bar{\pi}_t^i = \bar{\pi}_t^m, \; i = 1, ..., m-1, \; t \geqq 0$$

and the programme satisfies

$$\bar{p}_{t+1}\bar{y}_{t+1} - \bar{p}_t\bar{x}_t \geqq \bar{p}_{t+1}y - \bar{p}_t x, \quad \text{for all } (x, y) \in \mathscr{T} \quad \text{for } t \geqq 0. \qquad ...(2.2)$$

To see this, use the concavity of $f(\cdot)$ to write

$$(y^m - \bar{y}_{t+1}^m) \leqq f(^*y, x) - f(^*\bar{y}_{t+1}, \bar{x}_t) \leqq \sum_{i=1}^{m-1} \bar{f}_{y^i}^{t+1}(y^i - \bar{y}_{t+1}^i) + \sum_{i=1}^{m} \bar{f}_{x^i}^{t+1}(x^i - \bar{x}_t^i).$$

For $t = 0$, then, we have $\bar{p}_1(y - \bar{y}_1) < \bar{p}_0(x - \bar{x}_0)$.

For $t \geqq 1$, using (2.1), we have

$$(y^m - \bar{y}_{t+1}^m) - \sum_{i=1}^{m-1} \bar{f}_y^{t+1}(y^i - y_{t+1}^i) \leqq \bar{f}_{x^m}^{t+1} \sum_{i=1}^{m} (-\bar{f}_{y^i}^t)(x^i - \bar{x}_t^i),$$

and noting that

$$\bar{\pi}_t^i = \bar{\pi}_t^m i = 1, ..., m-1, \; \bar{p}_{t+1}(y - \bar{y}_{t+1}) \leqq \bar{p}_t(x - \bar{x}_t).$$

Therefore, for all $t \geqq 0$, (2.2) holds.

For the interior programme $(\bar{x}, \bar{y}, \bar{c})$, the ratio $(\bar{p}_1^i/\bar{p}_t^i)$ is to be interpreted as the terms of trade, from period 1 to period t, for good i. The terms of trade for good i along the programme are said to *deteriorate too fast* if the discounted price system $(\bar{p})$ satisfies

$$\sum_{t=1}^{\infty} (\bar{p}_1^i/\bar{p}_t^i) < \infty. \qquad ...(2.3)$$

3. A COMPLETE CHARACTERIZATION OF EFFICIENCY

We can now state our result. Detailed proofs will be given in the next section. Only the basic steps in the proof will be given here.

Theorem 3.1. *Under assumptions* (A.1)-(A.4), *an interior programme* $(\bar{x}, \bar{y}, \bar{c})$ *from* $\mathbf{x}$, *is long-run efficient if and only if* (i) *it is competitive and* (ii) *the terms of trade of none of the goods deteriorate too fast.*

Proof. (Sufficiency) Suppose $(\bar{x}, \bar{y}, \bar{c})$ is an interior programme, and it satisfies (i) and (ii) above, but is long-run inefficient. Then

Step 1. There exists a $t^* \geqq 1$, a scalar A, and (a) a sequence (θ_t) such that (1) $0 \leqq \theta_t \leqq A$ and (2) $\theta_{t+1} > \bar{f}_{x^m}^t \theta_t$ hold for $t \geqq t^*$: (b) a sequence (α_t) such that (1′) $0 < \alpha_t \leqq A$ and (2′) $\alpha_{t+1} \geqq (\bar{f}_{x^m}^t + \alpha_t)\alpha_t$ hold for all $t \geqq t^*$.

It follows from Step 1 that

Step 2. The terms of trade of the mth good deteriorate too fast. This contradicts condition (ii), and establishes sufficiency.

(Necessity) Suppose $(\bar{x}, \bar{y}, \bar{c})$ is an interior programme which is long-run efficient. Then we can show that

Step 3. It is competitive. This establishes condition (i).

To establish (ii), suppose it is violated. Then we can show that

Step 4. There is a sequence (β_t), such that

$$\text{(a) } 0 < \beta_t < \bar{x}_t^m \quad (b) \; \beta_{t+1} \geqq (\bar{f}_{x^m}^t + \beta_t)\beta_t \text{ hold for all } t \geqq 1.$$

It follows from Step 4 that

Step 5. We can construct a feasible programme (x, y, c) which dominates $(\bar{x}, \bar{y}, \bar{c})$, i.e. which has $c_t \geqq \bar{c}_t$ for all t, $c_t > \bar{c}_t$ for some t. This contradiction establishes (ii), and hence necessity.

4. PROOFS

(Sufficiency) *Step* 1 (a). Note that there exists a feasible programme (x', y', c') such that $c'_{t+1} \geqq \bar{c}_{t+1}$ for all $t \geqq 0$, $c'_{t+1} > \bar{c}_{t+1}$ for some t. Since $(\bar{x}, \bar{y}, \bar{c})$ is interior, $x^i_t \geqq k > 0$ for $t \geqq 0$, $i = 1, \ldots, m$. Construct the programme (x, y, c) as follows: $x_t = \frac{1}{2}\bar{x}_t + \frac{1}{2}x'_t$; $(x_t, y_{t+1}) \in \mathscr{T}$ such that $y_{t+1} \geqq \frac{1}{2}\bar{y}_{t+1} + \frac{1}{2}y'_{t+1}$; $c_{t+1} = y_{t+1} - x_{t+1}$ for $t \geqq 0$. By (A.1), (x, y, c) is feasible. Also, $c_{t+1} \geqq \frac{1}{2}\bar{c}_{t+1} + \frac{1}{2}c'_{t+1}$ for $t \geqq 0$, so $c_{t+1} \geqq \bar{c}_{t+1}$ for all $t \geqq 0$, $c_{t+1} > \bar{c}_{t+1}$ for some t. Finally, $x^i_t \geqq k/2 > 0$ for $t \geqq 0$, $i = 1, \ldots, m$.

Let t' be the first period for which $c_{t'+1} > \bar{c}_{t'+1}$. Since $(\bar{x}, \bar{y}, \bar{c})$ is interior and competitive, (2.2) holds. Using this property, for any $t \geqq 0$, we have

$$\Sigma_s p_{s+1}(c_{s+1} - \bar{c}_{s+1}) \leqq \bar{p}_{t+1}(\bar{x}_{t+1} - x_{t+1}), \text{ for } s = 0, \ldots, t$$

and for $t \geqq t'$ (since $\bar{p} = (\bar{p}_s)$ satisfies $\bar{p}_s \gg 0$ for $s \geqq 0$), $\bar{p}_{t+1}(\bar{x}_{t+1} - x_{t+1}) \geqq \xi > 0$. Also, since $(\bar{x}, \bar{y}, \bar{c})$ is interior and competitive, $\bar{\pi}^i_t = \bar{\pi}^m_t$, $t \geqq 0$, $i = 1, \ldots, m-1$. So

$$\bar{q}_{t+1}(\bar{x}_{t+1} - x_{t+1}) \geqq \xi \bar{\pi}^m_t > 0$$

for $t \geqq t'$. Thus for $t \geqq t'$, $\bar{x}_{t+1} \neq x_{t+1}$. So, by using the strict concavity of $f(\cdot)$ (in the steps used to prove (2.2)) we have $\bar{p}_{t+1}\bar{y}_{t+1} - \bar{p}_t\bar{x}_t > \bar{p}_{t+1}y_{t+1} - \bar{p}_t x_t$ for $t \geqq t'+1$. So,

$$\bar{p}_{t+1}(\bar{x}_{t+1} - x_{t+1}) = \bar{p}_{t+1}(\bar{y}_{t+1} - \bar{c}_{t+1}) - \bar{p}_{t+1}(y_{t+1} - c_{t+1})$$

$$= \bar{p}_{t+1}(\bar{y}_{t+1} - y_{t+1}) + \bar{p}_{t+1}(c_{t+1} - \bar{c}_{t+1}) > \bar{p}_t(\bar{x}_t - x_t)$$

for $t \geqq t'+1$, or,

$$\bar{q}_{t+1}(\bar{x}_{t+1} - x_{t+1})/\bar{\pi}^m_{t+1} > \bar{q}_t(\bar{x}_t - x_t)/\bar{\pi}^m_t,$$

so that $\bar{q}_{t+1}(\bar{x}_{t+1} - x_{t+1}) > \bar{f}^t_{x^m}\bar{q}_t(\bar{x}_t - x_t)$ for $t \geqq t'+1$. Define $\bar{q}_t(\bar{x}_t - x_t) = \theta_t$. Since $(\bar{x}, \bar{y}, \bar{c})$ is interior, and $\bar{x}^i_t$, $\bar{y}^i_{t+1} \leqq K$ for $i = 1, \ldots, m$, $t \geqq 0$ (by (A.4)), so $\bar{q}^i_t$ are uniformly bounded above, and there exists a scalar $A > 0$, such that $\theta_t \leqq \bar{q}_t\bar{x}_t \leqq A$. Defining $t^* = t'+1$, conditions (1) and (2) of Step 1 (a) hold for all $t \geqq t^*$.

Step 1 (b). For all $t \geqq t'+1$ in Step 1 (a), we have, by Taylor's expansion, $\theta_{t+1} = \bar{f}^t_{x^m}\theta_t + \frac{1}{2}Q_t$, where Q_t stands for the quadratic form $R_t' D_t R_t$,

$$R_t = ({}^*\bar{y}_{t+1} - {}^*y_{t+1}, \bar{x}_t - x_t) \quad \text{and} \quad D_t = (-\tilde{f}^{t+1}_{xy}).$$

The tilde ($\sim$) denotes that the second-order partial derivatives are evaluated at some point between $({}^*\bar{y}_{t+1}, \bar{x}_t)$ and $({}^*y_{t+1}, x_t)$. Then, by Step 1 (a), we have $(\theta_{t+1} - \bar{f}^t_{x^m}\theta_t)/\theta^2_t = Q_t/\theta^2_t > 0$, for all $t \geqq t'+1 = t^*$. We claim that there exists a scalar $\mu > 0$, such that $Q_t/\theta^2_t \geqq \mu$ for all $t \geqq t^*$. Suppose this is not true. Then there exists a subsequence t_s, such that $Q_{t_s}/\theta^2_{t_s} \to 0$ as $t_s \to \infty$. Consider the sequence of vectors (e_{t_s}) given by $(\bar{y}_{t_s+1}, y_{t_s+1}, \bar{x}_{t_s}, x_{t_s}, v_{t_s})$, where v_{t_s} denotes the vector of v's which determine the point at which $(\tilde{f}^t_{xy})$ is calculated in each case. Clearly (e_{t_s}) lies in a bounded region. So, it has a convergent subsequence, call it t_s again, such that e_{t_s} converges to e. Note that since both programmes are interior with a lower bound of $k/2$, so $(\tilde{f}^t_{xy})$ is always evaluated at some $(\tilde{x}, \tilde{y})$ such that $\tilde{x}^i$, $\tilde{y}^i \geqq k/2$ for $i = 1, \ldots, m$. By continuity of the partial derivatives, D_{t_s} converges to D, and D is positive definitive. Let $\beta_t = ({}^*\bar{y}_{t+1} - {}^*y_{t+1}, \bar{x}_t - x_t)/\theta_t$. Clearly $Q_t/\theta^2_t = \beta'_t D_t \beta_t$. We shall now show that $\beta_{t_s} \to 0$. Rename t_s as T. It is clear that if β_T is unbounded so is $\beta'_T D_T \beta_T$. Since $\beta'_T D_T \beta_T \to 0$, β_T must be bounded, and have a convergent subsequence (call it T_u now), such that $\beta_{T_u} \to \beta$. Since $Q_{T_u}/\theta^2_{T_u} = \beta'_{T_u} D_{T_u} \beta_{T_u}$, so taking limits on both sides, we have $0 = \beta' D \beta$, so that $\beta = 0$, since D is positive definite. Since this is true of any convergent subsequence of β_T, so $\beta_T \to 0$ as $T \to \infty$.

Now, along the subsequence T, let $z_T = \max_i | \bar{x}^i_T - x^i_T |$ $i = 1, \ldots, m$. Then z_T must equal $| \bar{x}^j_T - x^j_T |$ an infinite number of times, for some j in the set $(1, 2, \ldots, m)$. Call such a subsequence T_r. Then since $\beta_T \to 0$, so $\beta_{T_r} \to 0$. So, given any $\varepsilon > 0$, there exists a T^*, such that for $T_r \geqq T^*$, $(|\bar{x}^j_{T_r} - x^j_{T_r}|)/(|\bar{q}_{T_r}(\bar{x}_{T_r} - x_{T_r})|) < \varepsilon$. This implies that

$$|\bar{x}^j_{T_r} - x^j_{T_r}| < \varepsilon\, |\bar{x}^j_{T_r} - x^j_{T_r}|(\Sigma_i \bar{q}^i_{T_r}) \text{ for } i = 1, ..., m.$$

Also, $\Sigma_i \bar{q}^i_{T_r}$ is uniformly bounded above for all T_r by a scalar M. So choosing $\varepsilon = 1/M$, we have a contradiction, establishing our claim of a lower bound $\mu > 0$, such that $Q_t/\theta_t^2 \geqq \mu$ for all $t \geqq t^*$. Let $\bar{\mu} = \min(1, \mu/2)$ and let $\alpha_t = \bar{\mu}\theta_t$. Then we have a sequence (α_t) satisfying conditions (1′) and (2′) of Step 1 (b), for all $t \geqq t^*$.

Step 2. Follow the method of Cass [1, p. 219].

(Necessity) *Step* 3. $(\bar{x}, \bar{y}, \bar{c})$ is long-run efficient, hence, short-run efficient. For each T, $T \geqq 2$, $(\bar{x}, \bar{y}, \bar{c})$ solves the following non-linear programming problem: maximize $f(^*\bar{y}_{T+1}, x_T)$ subject to $c_{t+1} \geqq \bar{c}_{t+1}$, $t = 0, ..., T-1$, among all feasible programmes (x, y, c) from $\boldsymbol{x}$. To apply the Kuhn-Tucker theorem, we check that the constraint qualification is satisfied, i.e. there is a feasible programme for which each constraint holds with a strict inequality.

Since $(\bar{x}, \bar{y}, \bar{c})$ is interior, $k \leqq \bar{x}^i_t, \bar{y}^i_{t+1} \leqq K$, $t \geqq 0$, $i = 1, ..., m$. For t, such that $0 \leqq t \leqq T$, since from $\bar{x}_t$, output $\bar{y}_{t+1}$ is producible, so the output $\hat{y}_{t+1}$, given by $^*\bar{y}_{t+1} = {}^*\hat{y}_{t+1}$, and $0 < \hat{y}^m_{t+1} < \bar{y}^m_{t+1}$, is also producible, by (A.2). Also $\tilde{y}_{t+1} \gg 0$ is producible from $\tilde{x}_t$, where $^*\tilde{x}_t = {}^*\bar{x}_t$ and $\tilde{x}^m_t = \bar{x}^m_t - \varepsilon_t$, where $0 < \varepsilon_t < k/2$. By taking a suitable convex combination $(0 < \lambda_t < 1)$, from $\bar{\bar{x}}_t = [^*\bar{x}_t, \bar{x}^m_t - \lambda_t \varepsilon_t]$, we can produce $\bar{\bar{y}}_{t+1}$ given by $^*\bar{\bar{y}}_{t+1} = [^*\bar{y}_{t+1} + {}^*\eta_{t+1}]$ (where $^*\eta_{t+1} \gg 0$), and $\bar{\bar{y}}^m_{t+1} > 0$. Let $\delta_t = \lambda_t \varepsilon_t$, and $\delta = \min \delta_t$ for $0 \leqq t \leqq T$.

Since f_{x^m} is continuous on $[k/2, K]$, so there exists $M > 0$, such that $f_{x^m} \leqq M$, for $K \geqq (x^i, y^i) \geqq k/2$ for all i. Now, let $\mu_0 = 0$, $\mu_t = \min[\delta/(2M)^{T-t}, 1]$, $t = 1, ..., T$, and construct the required programme (x, y, c) as follows:

$$x_0 = \boldsymbol{x}, \text{ and for } t = 0, ..., T-1,$$

$$y_{t+1} = [^*\bar{\bar{y}}_{t+1}, \bar{\bar{y}}^m_{t+1} - \mu_{t+1}]; \quad x_{t+1} = [^*\bar{\bar{x}}_{t+1}, \bar{\bar{x}}^m_{t+1} - 2\mu_{t+1}]$$

and

$$c_{t+1} = [^*\bar{c}_{t+1} + {}^*\eta_{t+1}, \bar{c}^m_{t+1} + \mu_{t+1}]; \quad (x_{t+1}, y_{t+1}, c_{t+1}) = (0, 0, 0) \quad \text{for } t \geqq T.$$

To check feasibility, note that

(a) $(x_{t+1}, y_{t+1}, c_{t+1}) \gg 0 \quad t = 0, ..., T-1$

(b) $y_{t+1} = x_{t+1} + c_{t+1} \quad t = 0, ..., T-1$

(c) $(x_t, y_{t+1}) \in \mathscr{T} \quad t = 0, ..., T-1$.

To see this, note that since $\mu_t \leqq \delta \leqq \delta_t = \lambda_t \varepsilon_t$ so $f(^*y_{t+1}, x_t) > 0$ and

$$f(^*y_{t+1}, x_t) - f(^*\bar{\bar{y}}_{t+1}, \bar{\bar{x}}_t) = \tilde{f}^{t+1}_{x^m}(-2\mu_t) > -[\delta(2M)\mu_t]/[(2M)^{T-t}] \geqq -\mu_{t+1}.$$

So,

$$f(^*y_{t+1}, x_t) > f(^*\bar{\bar{y}}_{t+1}, \bar{\bar{x}}_t) - \mu_{t+1} = \bar{\bar{y}}^m_{t+1} - \mu_{t+1} = y^m_{t+1}.$$

It is clear that the constraints are satisfied with strict inequality, since $c_{t+1} \gg \bar{c}_{t+1}$ for $t = 0, ..., T-1$.

Now, applying the Kuhn-Tucker theorem, we know that $(\bar{x}, \bar{y}, \bar{c})$ must maximize the Lagrangean, $L(\cdot) = f(^*\bar{y}_{T+1}, x_T) + \Sigma_t \mu_{t+1}(c_{t+1} - \bar{c}_{t+1})$ for $t = 0, ..., T-1$ and for some choice of multipliers $\mu_{t+1} = (\mu^1_{t+1}, ..., \mu^m_{t+1})$. Since $(\bar{x}, \bar{y}, \bar{c})$ is interior, the necessary conditions of a maximum are:

$$\partial L/\partial x^m_t = \mu^m_{t+1}\bar{f}^{t+1}_{x^m} + \mu^m_t(-1) = 0; \quad \partial L/\partial x^i_t = \mu^m_{t+1}\bar{f}^{t+1}_{x^i} + \mu^m_t[+f^t_{y^i}] = 0 \quad i = 1, ..., m-1$$

(for $t = 1, ..., T-1$)

$$\partial L/\partial x^m_T = \bar{f}^{T+1}_{x^m} + \mu^m_T(-1) = 0; \quad \partial L/\partial x^i_T = \bar{f}^{T+1}_{x^i} + \mu^m_T[+\bar{f}^T_{y^i}] = 0.$$

Since the Lagrange multipliers are all seen to be positive from the above equations, we get the conditions:

$$\bar{f}_{x^i}^{t+1}/f_{x^m}^{t+1} = [-\bar{f}_{y^i}^t] \quad \text{for } t = 1, \ldots, T.$$

Now, suppose the competitive conditions (2.2) are violated for some $t = \tau \geqq 1$. Choose T, in the above exercise, to be $\tau+1$. We get an immediate contradiction. This establishes the competitive conditions (2.2) for all $t \geqq 1$.

Step 4. The terms of trade of some good deteriorates too fast. So, using the result of Step 3, and noting that, for the interior programme $(\bar{x}, \bar{y}, \bar{c})$, $\bar{q}_t^i$ $(i = 1, \ldots, m)$ is uniformly bounded above and bounded away from zero, we have that the terms of trade of good m must deteriorate too fast. Now follow the method of Cass [1, p. 220].

Step 5. First, we note that there exists a $\delta > 0$ (independent of t), and an output level $y_{t+1}^m > 0$, such that $y_{t+1}^m = f(^*\bar{y}_{t+1}, {}^*\bar{x}_t, \bar{x}_t^m - \delta)$ for all $t \geqq 1$. Since from $\bar{x}_t$, output $\bar{y}_{t+1}$ is producible, so the output $\hat{y}_{t+1}$ given by $^*\hat{y}_{t+1} \gg {}^*\bar{y}_{t+1}$ and $0 < \hat{y}_{t+1}^m < \bar{y}_{t+1}^m$ is also producible (by (A.2)). Also, $\tilde{y}_{t+1} \gg 0$ is producible from $\tilde{x}_t$, given by $^*\tilde{x}_t = {}^*\bar{x}_t$ and $\tilde{x}_t^m = \bar{x}_t^m - \varepsilon_t$, where $0 < \varepsilon_t < k/2$. By taking a suitable convex combination, from $\bar{\bar{x}}_t = (^*\bar{x}_t, \bar{x}_t^m - \lambda_t \varepsilon_t)$, one can produce $\bar{\bar{y}}_{t+1}$, given by $^*\bar{\bar{y}}_{t+1} = {}^*\bar{y}_{t+1}$, and $\bar{\bar{y}}_{t+1}^m > 0$. Let $\delta_t = \sup \lambda_t \varepsilon_t$, such that the above construction is valid. This establishes a δ_t for each t. We want a $\delta > 0$, uniform for all $t \geqq 1$, such that the construction is valid. Suppose there is no such δ. Then there exists a subsequence t_s such that $\delta_{t_s} \to 0$. Note that we have $\bar{\bar{y}}_{t+1}^m - \bar{y}_{t+1}^m = \tilde{\bar{f}}_{x^m}^t(-\delta_t)$. And, $\tilde{\bar{f}}_{x^m}^t$ is bounded above by a scalar N. So for $\delta_{t_s} \leqq k/2N$, we have $\bar{\bar{y}}_{t_s+1}^m \geqq k/2$. But then we could apply the original procedure to $(\bar{\bar{x}}_{t_s}, \bar{\bar{y}}_{t_s+1}^m)$, and get $\gamma_{t_s} > 0$, such that from $x_{t_s} = (^*\bar{x}_{t_s}, \bar{x}_{t_s}^m - \delta_{t_s} - \gamma_{t_s})$ one can produce $y_{t_s+1} = (^*\bar{y}_{t_s+1}, y_{t_s+1}^m)$, where $y_{t_s+1}^m > 0$. This would contradict the fact that $\delta_{t_s} = \sup \lambda_{t_s} \varepsilon_{t_s}$, and establish the uniform δ.

Now, we can construct a dominating programme as follows. Consider the sequence $(\hat{x}_t)$ such that $x_t^m = \bar{x}_t^m - \beta_t/2$, $^*\hat{x}_t = {}^*\bar{x}_t$. Then $\hat{x}_t^i \geqq k/2$, $i = 1, \ldots, m$, for all $t \geqq 0$. Now, $(-f_{x^m x^m})$ is continuous on the positive orthant, so it is bounded above by a scalar B, if it is evaluated at any convex combination of $(\bar{x}_t, {}^*\bar{y}_{t+1})$ and $(\hat{x}_t, {}^*\bar{y}_{t+1})$. Since $(\bar{x}, \bar{y}, \bar{c})$ is interior, there exists a $\delta > 0$ (established above) satisfying

$$y_{t+1}^m > 0, \quad \text{and} \quad y_{t+1}^m = f(^*\bar{y}_{t+1}, {}^*\bar{x}_t, \bar{x}_t^m - \delta)$$

for all $t \geqq 1$. Choose $\lambda = \min(1/2, B/2, \delta/K)$ and consider the programme (x, y, c) such that $^*x_t = {}^*\bar{x}_t$, $x_t^m = \bar{x}_t^m - \lambda\beta_t$ for all $t \geqq 1$, and $^*c_1 = {}^*\bar{c}_1$, $c_1^m = \bar{c}_1^m + \lambda\beta_1$, $c_t \geqq \bar{c}_t$ for all $t \geqq 2$.

We claim that this programme is feasible. First, note that it is feasible for $t = 1$. We shall show that if it is feasible upto T, then it is feasible upto $T+1$. For period T, using the fact that $\lambda \leqq \delta/K$, we have $x_T^m \geqq \bar{x}_T^m - \delta$. By construction of δ, there exists an output $y_{T+1}^m > 0$, such that $y_{T+1}^m = f(^*\bar{y}_{T+1}, {}^*\bar{x}_T, \bar{x}_T^m - \delta)$. Now, suppose that

$$y_{T+1}^m < \bar{y}_{T+1}^m + \lambda\beta_{T+1}.$$

Then we have

$$\lambda\beta_{T+1} < f(^*\bar{y}_{T+1}, \bar{x}_T) - f(^*\bar{y}_{T+1}, {}^*\bar{x}_T, \bar{x}_T^m - \lambda\beta_T) < \lambda f_{x^m}^T \beta_T.$$

This implies that $(\beta_{T+1} - f_{x^m}^T \beta_T)/\beta_T^2 < 0$. Also, note that

$$(\beta_{T+1} - \bar{f}_{x^m}^T \beta_T)/\beta_T^2 - (\beta_{T+1} - f_{x^m}^T \beta_T)/\beta_T^2 = (f_{x^m}^T - \bar{f}_{x^m}^T)/\beta_T = (-\bar{f}_{x^m x^m}^T)\lambda < \beta\lambda \leqq \tfrac{1}{2}.$$

But since $(\beta_{T+1} - \bar{f}_{x^m}^T \beta_T)/\beta_T^2 \geqq 1$, so we have the contradiction

$$\tfrac{1}{2} = 1 - \tfrac{1}{2} \leqq (\beta_{T+1} - \bar{f}_{x^m}^T \beta_T)/\beta_T^2 < 0.$$

This proves that $y_{T+1}^m \geqq \bar{y}_{T+1}^m - \lambda\beta_{T+1}$. Thus the programme is feasible upto $T+1$. This completes the induction step, and shows that (x, y, c) is feasible. Also, by construction, (x, y, c) dominates $(\bar{x}, \bar{y}, \bar{c})$.

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